Generalizing chess to higher dimensions

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Introduction

Chess—the 2-player game famously played on a 2-dimensional, eight-by-eight grid of squares—is a beloved and incredibly long-standing game. But what happens if chess has become *boring*? The most obvious answer is to raise the game of chess to higher dimensions, but how does one go about such a task? The idea of chess in higher dimensions is not a purely original one, nor is it one that can be traced back to any one person. Mathematicians and various thinkers often like to experiment with concepts of higher dimensions, and chess, being a well-known, easy-to-learn and hard-to-master game, seems to be popular candidate for such experimentation.

The goal of this paper is to generalize the rules and properties of chess to higher dimensions and define a ruleset that is consistent across any game of n -dimensional chess for $n \geq 2$. This ruleset should behave identically to the standard rules of chess when applied to a traditional (2-dimensional, 8-by-8) chessboard. This paper largely focus on chess in 4 dimensions, however, all concepts can be applied to any game of any finitely large game of chess.

This paper is not intended to be a guide for playing higher-dimensional chess, but instead a concretely defined set of rules that can be used as a basis for guides or more complex n-dimensional chess ideas.

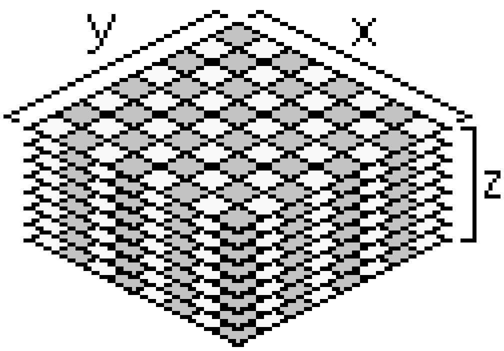
Understanding 3-dimensional chess

Because our reality exists in 3-dimensions, raising chess to higher dimensions (4+) means thinking of it not as a physical board game that exists in space, but as a concept which can be represented by numbers on paper and projections into physical space. In some cases, the 4th dimension is considered time. In this paper, such is not the case, as we are referring to higher spatial dimensions. These higher spatial dimensions do not physically exist, but we can project them onto 2 dimensional space.
We must understand how we might represent chess in the 3rd dimension, before raising it to higher dimensions.
Traditional chess is a game in two dimensions, as any possible movement of

pieces exists exclusively along 2 axes. In chess, the axes are called rank (1-8) and file (a-h). Alternate names for the axes include: Horizontal and vertical, forward-and-back and side-to-side, and x and y. In this paper we will use x and y.

To imagine more than 3 dimensions, we first have to conceptualize 3-dimensional chess, which can be thought of as 8 stacked chessboards, or an 8-by-8-by-8 chess-cube. Physically stacking chessboards gives the game a third (z) axis, which pieces may be moved along. For example: a rook could move up or down in a straight line, staying at the same square on any chessboard (same x and y coordinates), but moving to a different board in the stack (different z coordinate).

Example of 8-by-8-by-8, 3-dimensional, stacked board, with labelled axes.



Playing 3-dimensional chess in 3 dimensions by stacking chessboards would prove to be an impractical task, as one would have to vertically space the chessboards as to give room for players to move their pieces. In higher dimensions, this problem worsens, making it not impractical to physically represent every dimension, but entirely impossible.

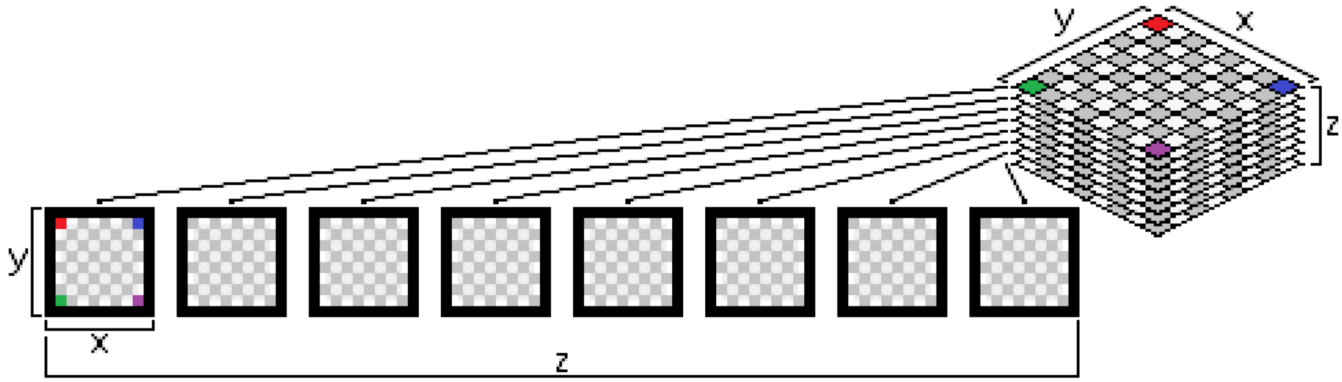
Projection into 3 dimensions

To make games of higher dimensional chess possible, we must project this higher dimensional game onto 2 dimensions, similar to how a globe is projected on to a map, or how our 3-dimensional world is projected through a camera's lens onto a 2-dimensional image.

We start by considering the projection of the 8-by-8-by-8 chess-cube. Instead of having 8 physical chessboards vertically stacked, we take each chessboard and place them in a line, now having 8 chessboards, each 8-by-8 themselves, in a row.

There is an important distinction to be made between board and chessboard. We say that the space (conceptual or physical) which any game of n-dimensional chess is played on is the board; in describing projections, we will use chessboard to describe a physical, 2-dimensional board. This distinction is made as we are imagining the entire board as a stack of physical chessboards.

Below is an example of a 3-dimensional "stacked" board being projected into 2 dimensions. Lines and colored corners exist as visual aid.



With this projection, "vertical" movement of pieces along the z axis now is no longer *physically* represented with vertical movement, but is represented by movement between chessboards.

It is important to note that in a projection from 3 dimensions to 2 dimensions—as if the game was being played on a table—moving a piece along the x or z axis now appears similar, as they both mean moving a piece side-to-side, although one is within a single chessboard, and the other is across chessboards. This is a problem which may cause significant confusion initially; it must be understood that despite having some similarity in appearance, movement along the board's axes is unchanged. x, y, and z still act as they would on a stacked board, with the only change being how it is visualized.

Conceptualizing higher dimensions

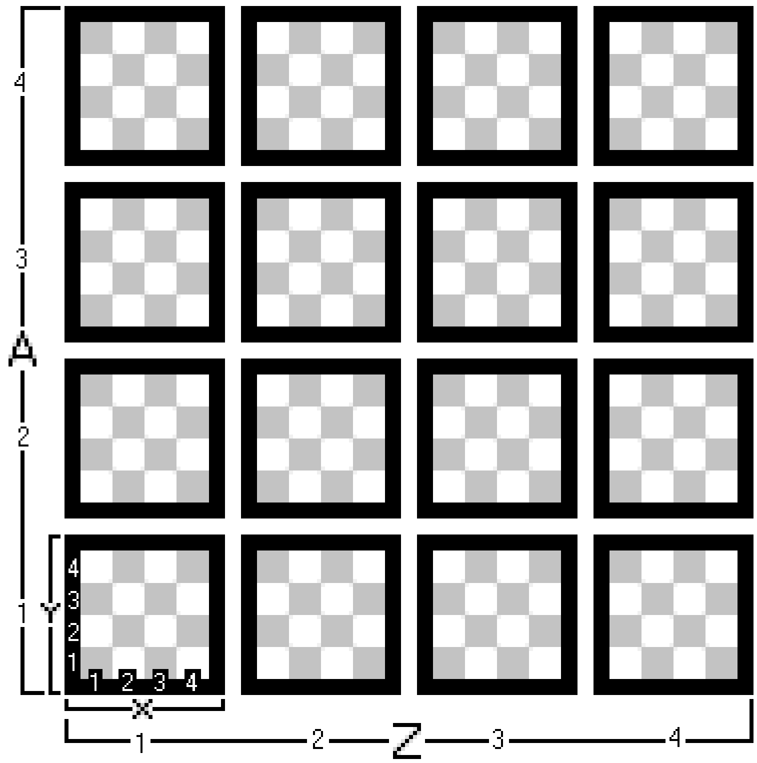
Now that we understand projecting a 3-dimensional game into 2-dimensions, we can consider how we might project 4 or more dimensions into 2-dimensions.

The same idea of projection continues for higher dimensions. For games of chess in the 4th dimension, it's impossible to physically visualize or realize a 4-dimensional board, so we will think of higher dimensional chess purely as the board's projection into 2 dimensions.

To project any higher-dimensional board onto 2 dimensions, we forgo the process of imagining it as a physical object that could exist, instead imagining it purely as a projection.

It is worth noting that for 4-dimensions, it is useful to reduce the size of a board along each axis only to make visualizations easier, as a 2-dimensional projection of a 4-dimensional, 8-by-8-by-8-by-8 board is quite large (4,096 spaces) and impractical to draw. With each added dimension, the projections get exponentially larger.

We can now consider a 4-dimensional, 4-by-4-by-4-by-4 board. Below is an example of the projection of this board with axes labelled and numbered.



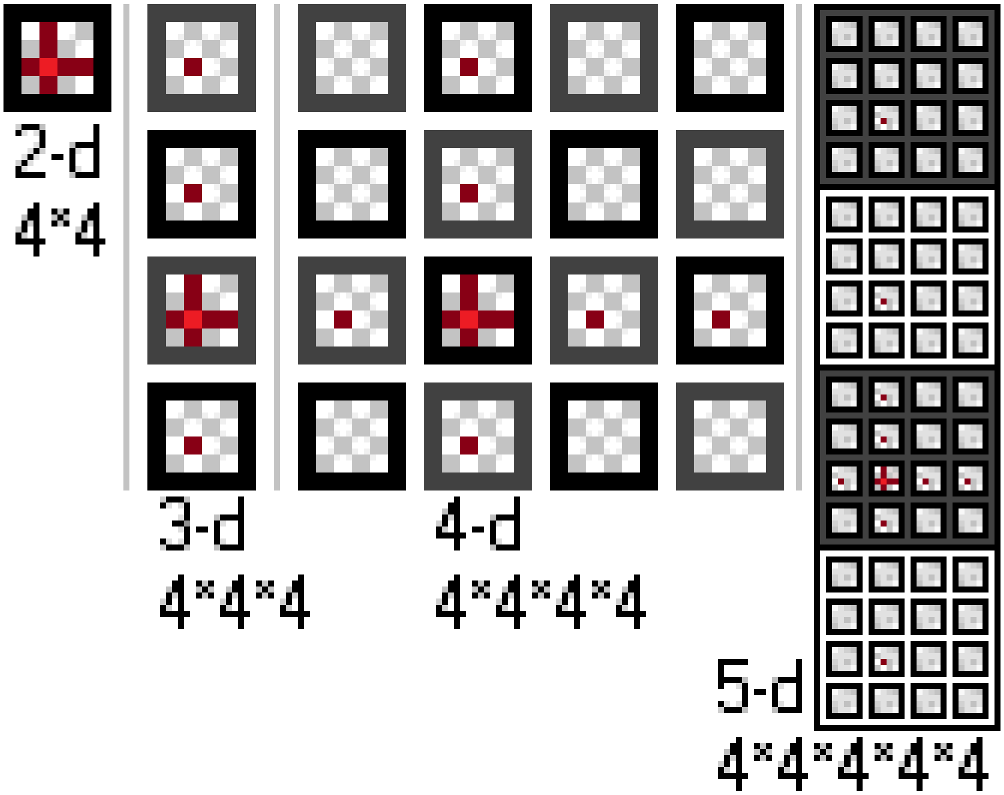
Comparing this projection to the 3-dimensional board's projection, we immediately notice how it appears not as a line of physical chessboards, but as a grid. This is due to the inclusion of the 4th axis a. The problem with the 3-dimensional board's projection where x and z appear similar continues to the 4-dimensional board's projection, worsening with the inclusion of a, which appears similar to y. Once again, the fact that x and z or a and y appear similar is a purely a side-effect of projection.

For a piece to move along the a axis, its position along the x, y, and z axes stay unchanged, only changing its position along the a axis.

Projections from higher dimensions work similarly. For 5 dimensions, we imagine taking 4 copies of the 4-dimensional projection shown above and placing them in a line, similar to the process of projecting a 3-dimensional cube into 2 dimensions (see "5-d" in example).

Below is an example of 5 projections of boards from 2 to 5 dimensions. Instead of projecting the board as chessboards in a row, this example is projected into columns. This difference is purely visual, as if the previous projections were rotated 90 degrees. To aid in understanding, we show how a rook could move across each of these boards. Highlighted in bright red is the position of a rook.

Every space that the rook may move to on its turn is colored dark red.



Notation

Firstly, we must define a new set of notation for playing and discussing any aspect of n-dimensional chess.

The variable d
We use d to refer to the number of dimensions (axes) a board has.
For a traditional (eight-by-eight, 2-dimensional) chessboard, there are two axes: rank (row) and file (column), so we say that $d = 2$.

Labelling of axes
In this paper, all axes will be labelled starting with x, y, and z, then wrapping around the alphabet back to a, b, c, and so on. This is done to follow the convention of using x, y and z for 2 and 3-dimensional coordinate systems.
The way in which any individual or group labels the axes is entirely up to preference. Pure alphabetical labelling ($a, b, c \ldots$) may be preferable for some, and for boards where $d > 26$, numbered labelling ($a_1, a_2, a_3 \ldots$ becomes useful.

Position
Instead of algebraic (a1, d5, etc.) notation, we will refer to any space on the board as a point, using a set of coordinates (x, y). In games where $d > 2$, we include d comma separated coordinates. For example: in a game where $d = 4$, every space is represented by 4 coordinates (x, y, z, a).

Board
We refer to the collection of spaces that any game of n-dimensional chess is played on as a board, despite the term only being technically correct when $d = 2$ (board in this sense refers to an object with a large flat surface on which a game is played).
Since we are no longer limited to any one size of board, notation for specific boards is required. Typically, we refer to the size of a 2-dimensional grid (or square) with the expression $x * y$, often written "x×y" or spoken as "x-by-y," where x and y correspond to the length of each side along an axis. The product of said expression is equal to the grid's total area.
We will refer to any board as "board $\{x * y * z * a...\}$". A standard chessboard becomes board $\{8 * 8\}$. A 5-dimensional, 4-by-4-by-4-by-3-by-3 board becomes board $\{4 * 4 * 4 * 3 * 3\}$.
We can then use exponentiation for boards where 2 or more of its axes share a length to shorten the notation further. Board $\{8 * 8\}$ becomes board $\{8^2\}$, board $\{4 * 4 * 4 * 3 * 3\}$ becomes board $\{4^3 * 3^2\}$. The expression $4^3 * 3^2$ is equal to the number of spaces in board $\{4^3 * 3^2\}$.

Note: "board $\{8^2\}$" will be used in place of "standard chessboard" or "traditional chessboard" when applicable.

Allowed piece movement
To describe the set of moves that a piece is allowed to make, we use the notation nA, where n is the number of spaces a piece moves along some axis A. If a piece is allowed to move some arbitrary number of spaces, n is written. Otherwise, n is replaced with the specific allowed number of spaces. n may be any value so long as it does not move a piece outside of the board's bounds. We say that nA is a single term.
For pieces with more complex movement, it is required to use multiple terms and multiple axes. In such a case, we use A_1 and A_2, and comma separate each term (Ex. $2A_1, 1A_2$). A_1 and A_2 may be any signed axis ($+y$, $-x$, etc.), but cannot be the same axis, regardless of sign. Additionally, within a single move, n must be consistent across terms; if n occurs more than once (Ex. move: nA_1, nA_2), it must be the same number across all terms of a move. Additional axes A_3, A_4, etc. may be used, but no current pieces require more than two simultaneous axes of movement.
This notation may also be used for referring to the distance between two spaces. For example: (1,1,1,1) → (1,2,4,1) is a distance of $1A_1$, $3A_2$. The number of terms refers to the spaces' dimensional distance, or how many axes along which they do not share a coordinate. For example: (1,1,1,1) → (2,4,3,3) is a distance of $1A_1, 3A_2, 2A_3, 2A_4$, which has 4 terms (a dimensional

distance of 4), meaning that if a piece wishes to move between those spaces, it must move across 4 axes.

Pieces

Pieces are labelled cP_n, where c is color (b or w), P is the standard single-letter piece name, and n is a number assigned to any piece with same-color multiples. n is omitted if the piece has no same-color multiples.

The number n is assigned to pieces by a reflected lexicographic ordering of their coordinates. When $d = 4$, this means that we sort the pieces by lowest a, then z, then y, then x.For example: if the pieces have been set up such that there is a white rook at (1,1,1,1) with duplicates 1 space away on the x, y, z, and a axes, they are labelled as such:

wR_1 (1,1,1,1), wR_2 (2,1,1,1), wR_3 (1,2,1,1), wR_4 (1,1,2,1), wR_5 (1,1,1,2)

In the case of promotion, a piece is given the lowest number n which is not already used by any piece of its same type and color.

Recording moves

Traditionally, to record the moves within a game of chess between two players, algebraic notation is used. This form of notation works for standard 2-dimensional chess, but not higher dimensions, therefore we must define notation for the moves of a game of any dimensional size.

We will say that any move within a game is written: piece, action, destination.

For example: the notation for white's pawn #1 on (2,2,1,1) advancing 2 spaces to (2,4,1,1) is wP_1 (2,4,1,1). Moving is implied and does not count as an action; if a piece moves and does nothing else, the action section is omitted.

All actions are as follows:

- Capturing: x cP_n
  - Example: wQ x bQ (3,4,2,2)
- En Passant: ep cP_n
  - Example: wP_1 ep bP_2 (3,3,1,3)
- Castling: 0-0 cR_n
  - Example: wK 0-0 wR_2 (3,2,4,1)
  - Although two pieces move, we list castling as the king's move and include the rook as part of the action. The coordinates are that of the king's destination, not the rook's.
- Promotion: = cP_n

- Example: wP_1 = wQ_2 (2,4,1,1)

The Board

Secondly, we must generalize the board's properties. Higher dimensions make generalizing certain elements of chess more difficult. When the board has 4 edges and 2 axes ($\{n^2\}$), the position of the players and what it means for a piece to move forward are significantly easier to describe than in higher dimensions.

Ends and sides

One property of a standard chessboard (board $\{8^2\}$) which is obvious and largely irrelevant to discuss is that it has exactly 2 opposite ends, one assigned to each player (black and white). This is not a particularly exciting property of chess, but it is one of the game's most important properties. The movement and promotion of pawns depend heavily on this property. If we were creating a new ruleset with purely compelling gameplay in mind, we may alter the movement rules of the pawn, number of players, and so on, but that would not be a proper generalization.

First we examine board $\{8^2\}$. Board $\{8^2\}$ has 2 sides, 2 ends, and 2 axes, x and y ($d = 2$).

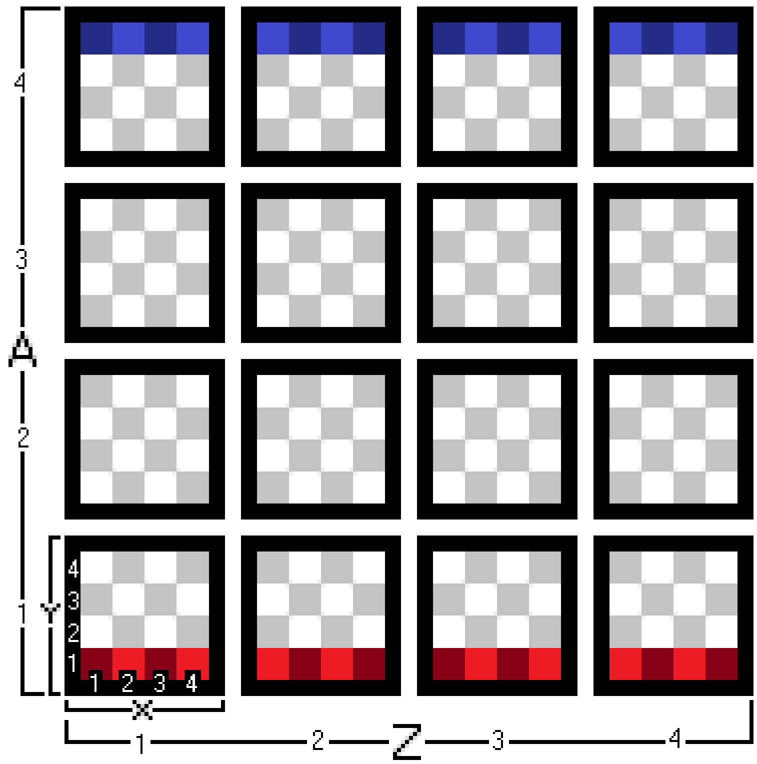
x runs parallel to the board's ends and connects the board's 2 sides, and y runs parallel to the board's sides and connects the board's 2 ends, so we say that x is the side axis (S) and y is the end axis (E). White's end of the board exists at the lowest point along y ($y = 1$), and likewise, black's end of the board exists at the highest point along y ($y = 8$), opposite from white's end.

Board $\{8^2\}$ has 1 side axis, 1 end axis, and 2 total axes; the ratio of end axes to total axes ($\frac{E}{d}$) is $\frac{1}{2}$, and end axes to side axes ($\frac{E}{S}$) is $\frac{1}{1}$.

We generalize these properties by saying that:

- Every n-dimensional board has exactly 2 opposite ends.
- Every second axis is an end axis so that the number of end axes is always half of the largest even number $n \leq d$. (For odd number d, $\frac{E}{S}$ approaches but does not equal $\frac{1}{1}$).
- White's end of the board exists where there is no lower point along any end axis.
- Black's end of the board exists where there is no higher point along any end axis.
- A piece has reached the opposite end of the board when it cannot move any farther from its color's end along any end axis.

Below is an example showing the ends of board $\{4^4\}$, whose 2 side axes are x and z, and end axes are y and a. The spaces colored blue represent black's end of the board, and the spaces colored red represent red's end of the board.



For any space along white's end (red), the y and a coordinates for that space are both 1, the minimum possible coordinate value. For any space along black's end (blue), the y and a coordinates for that space are both 4, the maximum possible value that a coordinate may be on board $\{4^4\}$. Reiterating in terms of coordinate notation, on board $\{4^4\}$, (x,1,z,1) is white's end and (x,4,z,4) is black's end, for any values x and z.

Defining forward

Carefully defining ends and sides was necessary to define forward movement, which is necessary for describing the pawn's rules of movement. We define forward as moving along some end axis E, such that it increases the distance along E between itself and the end of the board assigned to its color. A white piece moves forward along any $+E$, a black piece moves forward along any $-E$. For example: on board $\{n^4\}$, a white piece moves forward if, after moving, either its y or a coordinate increases.

(2,2,2,2) → (2,3,2,2) or (2,2,2,4) is forward motion, (2,2,2,2) → (3,2,2,2) or (2,1,2,2) is not forward motion.

Defining lines

For certain moves, we must define what it means for two pieces or spaces to be in a line from one another. That line may be either straight or diagonal.

For two spaces to be in a straight line, they must have a distance of nA, meaning they share all but one coordinate and have a dimensional distance of 1. The axis where they do not share a coordinate is the axis on which they are in a line. For example: (1,1,1,1) → (1,4,1,1) is a straight line along y with a distance of $3A$. (2,1,1,1) --> (3,1,2,1) is not a line and has a distance of $1A_1, 1A_2$.
For two spaces to be in a diagonal line, they share all but two of their coordinates. However, the coordinates which they do not share must differ by the same amount. The distance between two pieces in a diagonal line is always nA_1, nA_2. For example: (1,1,1,1) → (3,3,1,1) is a diagonal line along x and y with a distance of $2A_1, 2A_2$. (1,1,1,1) → (3,4,1,1) is not a line and has a distance of $2A_1, 3A_2$ (A_1 and A_2 have different distances, therefore it is not a line).

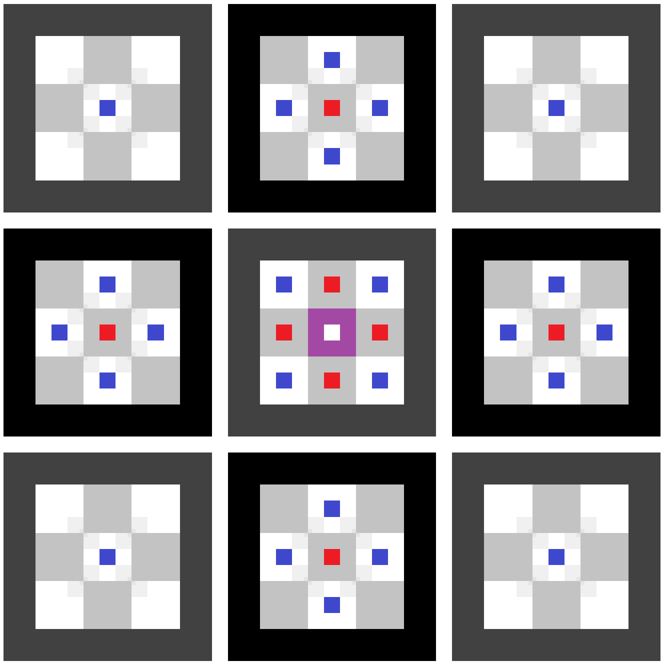
Setup

In traditional chess, the pieces are arranged in a consistent yet completely arbitrary way. One that was developed over a very long time. This means that defining rules which properly generalize the positions of pieces to boards other than $\{8^2\}$ becomes difficult and impractical. It is suggested instead that each group or individual decide on their own arrangement of pieces that best suits their board and playing style.

Adjacent opposite colors

If a higher dimensional board's spaces are colored in a black and white tiling, the property of orthogonally adjacent spaces being opposite colors and diagonally adjacent squares being same colors persists. The related property of bishops being only able to access squares whose color matches that of the bishop's starting square also persists.

Example on board $\{3^4\}$:



- White space (2,2,2,2) is marked purple.
- Orthogonally adjacent ($1A_1$ away) spaces are marked red.
- Diagonally adjacent ($1A_1, 1A_2$ away) spaces are marked blue.

Pieces

Finally, we must define the specific moves that pieces can make.
All moves are as follows:

Rook

- In a straight line: nA

Bishop

- In a diagonal line: nA_1, nA_2

Queen

- In a straight line: nA
- In a diagonal line: nA_1, nA_2

The rook, bishop, and queen may only move to a space if all spaces in a line between their current position and desired destination are vacant.

Knight

- Jump: $2A_1, 1A_1$
  - Ignores any pieces between its current position and destination.

King

- To any orthogonally adjacent space: $1A$
- To any diagonally adjacent space: $1A_1, 1A_2$

Castling
Conditions:

- The king and rook must not have previously moved, be the same color, and be in a straight line.
- There must be no pieces in the line between them.
- The king may not be currently in check or pass through a space which is under attack.

Execution:

- Let L be the axis along which both pieces do not share a common coordinate.
- The king moves $2L$ towards the rook.
- The rook moves nL towards the king, where n is their distance D $+1$.

Example on board $\{8^2\}$:

- King moves $2L$ towards rook: wK (5,1) → (7,1)
- Their distance (D), (7,1) to (8,1) is $1L$.
- Rook moves $D + 1$ $(2L)$ towards king: wR_2 (8,1) → (6,1)

Pawn
Let F be forward along any end axis and S be along any side axis.
The pawn may only capture diagonally $(1F, 1S)$.

- At any time, $1F$
- If the pawn has not yet moved, $2F$
- If the pawn is capturing, $1F, 1S$

En Passant

- If a pawn moves $2F$ and ends its turn orthogonally adjacent to an opponents pawn along a side axis, it may be captured by the opponent's pawn as though it had only moved $1F$.

- The capturing pawn moves $1F, 1S$.
- This is only legal on the opponent's move immediately after the pawn advances.

Promotion

- If a pawn reaches the end of the board assigned to the color opposite itself, it must promote into a queen, bishop, rook, or knight.

Conclusion

The set of rules which has been defined in this paper apply to any game of finitely large n-dimensional chess for $n \geq 2$. Some standard rules of chess, such as win conditions or turn order, are already consistent across all sizes of game, and therefore do not need to be included.